Algebraic *Analytic Urns*

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Outline

1. Analytic combinatorics on Pólya urns
2. Automatic search and proof
3. Classification

1. Analytic combinatorics on Pólya urns

Balanced Pólya urns

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \alpha, \delta \in \mathbb{Z}, \qquad \beta, \gamma \in \mathbb{N}$$

Balanced urn : $\boxed{\alpha + \beta = \gamma + \delta}$ (deterministic total number of balls)

A given initial configuration (a_0, b_0) :

a_0 balls $\bullet$ (counted by x)

b_0 balls $\circ$ (counted by y)

Definition

History of length n : a sequence of n evolutions (n rules, n drawings)

$$H(x, y, z) = \sum_{n,a,b} H_{n,a,b}\, x^a y^b \frac{z^n}{n!}$$

$H_{n,a,b}$: number of histories of length n, beginning in the configuration (a_0, b_0), and ending in (a, b).

Analytic properties

EDP [FlGaPe05]

$$\partial_z H = x^{a+1} y^b \, \partial_x H + x^c y^{d+1} \, \partial_y H \, .$$

Isomorphism theorem [FlDuPu06]

$$\text{Urn } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } \left\{ \begin{array}{c} (a_0, b_0) \\ \alpha + \beta = \gamma + \delta \end{array} \right. \Longrightarrow \begin{array}{c} H = X^{a_0} \, Y^{b_0} \\ \text{with } \left\{ \begin{array}{rcl} \dot{X} & = & X^{\alpha+1} \, Y^{\beta} \\ \dot{Y} & = & X^{\gamma} \, Y^{\delta+1} \end{array} \right. \end{array}$$

First integral for balanced urns [FlDuPu06]

Let $p := \gamma - \alpha = \beta - \delta$,

$$X^p - Y^p = x^p - y^p$$

Algebraicity Problem

$$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix} \qquad \begin{pmatrix}1 & 5\\4 & 2\end{pmatrix} \qquad \begin{pmatrix}5 & 2\\2 & 5\end{pmatrix} \qquad \begin{pmatrix}0 & 2\\1 & 1\end{pmatrix}$$

$$\begin{pmatrix}4 & 3\\2 & 5\end{pmatrix} \qquad \begin{pmatrix}1 & 4\\4 & 1\end{pmatrix} \qquad \begin{pmatrix}2 & 5\\0 & 7\end{pmatrix} \qquad \begin{pmatrix}3 & 1\\2 & 2\end{pmatrix}$$

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \qquad \begin{pmatrix}1 & 3\\0 & 4\end{pmatrix} \qquad \begin{pmatrix}2 & 2\\2 & 2\end{pmatrix} \qquad \begin{pmatrix}3 & 2\\2 & 3\end{pmatrix}$$

When is $H(x, y, z)$ algebraic ?

$$\begin{pmatrix}3 & 1\\1 & 3\end{pmatrix} \qquad \begin{pmatrix}4 & 5\\2 & 7\end{pmatrix} \qquad \begin{pmatrix}4 & 1\\1 & 4\end{pmatrix}$$

$$\begin{pmatrix}0 & 3\\2 & 1\end{pmatrix} \qquad \begin{pmatrix}4 & 3\\4 & 3\end{pmatrix} \qquad \begin{pmatrix}6 & 2\\3 & 5\end{pmatrix} \qquad \begin{pmatrix}4 & 1\\2 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \begin{pmatrix}2 & 1\\0 & 3\end{pmatrix} \qquad \begin{pmatrix}3 & 5\\6 & 2\end{pmatrix}$$

Algebraicity Problem

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2. Automatic search and proof

Automatic guess-and-prove

1. Power series expansion (150 terms)

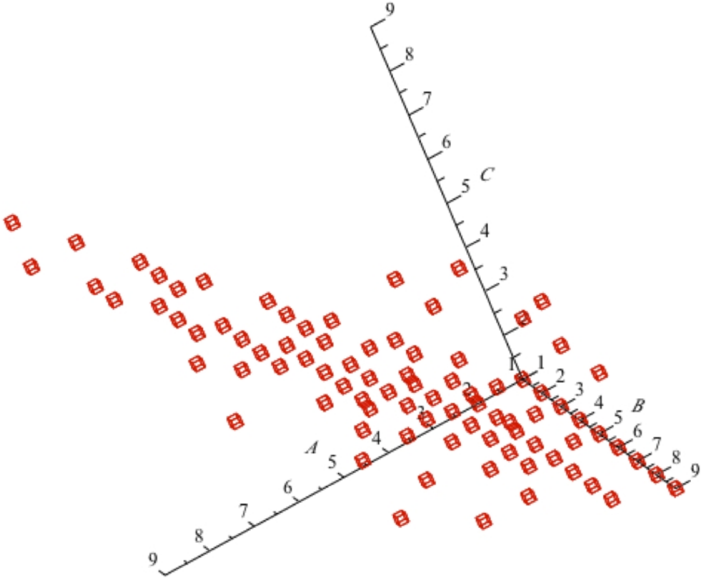
2. Guess an algebraic equation

3. Rigorous computer-aided proof

MAPLE Code

```
                %%% INITIALIZATION %%%
restart; libname := "//Users/basilemorcrette/", libname: with(gfun): gfun:-version();
pas := 150;
balancemax := 10;
CondInit := x;
                %%% CONSTRUCTION DE LA SERIE TRONQUEE %%%
Dop := proc(f,a,b,c,d)
expand( x^(1+a) * y^(b) * diff(f,x) + x^(c) * y^(1+d) * diff(f,y) ); end:

SerieCons := proc(n,init,a,b,c,d) local res, iter, i; option remember;
res := init; iter := init;
for i from 1 to n do
    iter := 1/i * Dop(iter,a,b,c,d);
    res := res+iter*z^i;
end: end proc:

                %%% AUTOMATIC GUESS-AND-PROVE %%%
for s from 1 to balancemax do
for a from 0 to s-1 do
for d from 1 to s-1 do
    b := s - a; c := s - d;
    if gcd(gcd(gcd(a,b),c),d)=1 then

                        %%% POWER SERIE EXPANSION %%%
        maserieenz := series(subs(y=1, SerieCons(pas, CondInit, a, b, c, d)), z, pas):

                        %%% GUESSING %%%
        tt := seriestoalgeq(maserieenz, h(z), [ogf]):

        if tt = 'FAIL' then
            print(Matrix([[a,b], [c,d]]), tt);
        else
                        %%% FORMAL PROOF %%%
            P := subs(h(z)=T, tt[1]):
            psi := RootOf(P,T):
            preuve := simplify(normal( (1-(a+b)*z*x^c) * diff(psi,z) + (x^(c+1) - x^(a+1)) * diff(psi,x) - x^c*psi));
            print(Matrix([[a,b],[c,d]]), P, preuve);
        end if:
    end if:
end: end: end;
```

Results



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ with } D = A + B - C.$$

3. Classification

Our classification theorem [M. et al. 2012]

Theorem: For an urn $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d > 0$, and $p := c - a = b - d$, if

(i) $p = 0$

(ii) $p < 0$ and $a \equiv 0 \ [p]$

(iii) $p \geq 2$, $a \equiv 1 \ [p]$ and $b \equiv -1 \ [p]$

then $H(x, y, z)$ is algebraic.

Recall: $H = X^{a_0} \ Y^{b_0}$ and $X^p - Y^p = x^p - y^p$.

Degree of minimal polynomial for Y

(i) $\sigma = a + b$

(ii) $\sigma = a + b$

(iii) $p(\sigma - p)$

Preliminary cases [FlDuPu06]

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ (Black $\leftrightarrow$ White)
- $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \sigma \\ c & d \end{pmatrix}$, with $\sigma, c > 0$ are not algebraic
- Any triangular urn $\begin{pmatrix} a & \sigma - a \\ 0 & \sigma \end{pmatrix}$, with $\sigma \geq a > 0$ or $\sigma > a \geq 0$ is algebraic.

Case (i) : $p = 0$

$$\begin{pmatrix} a & b \\ a & b \end{pmatrix}, \text{ with } a, b > 0 \ .$$

$$\begin{aligned} H(x,y,z) &= X(x,y,z)^{a_0} \qquad Y(x,y,z)^{b_0} \\ &= \left(\frac{x}{(1-\sigma x^a y^b z)^{1/\sigma}}\right)^{a_0} \left(\frac{y}{(1-\sigma x^a y^b z)^{1/\sigma}}\right)^{b_0} \end{aligned}$$

Y cancels the polynomial $\left(1 - \sigma x^a y^b z\right) \ Y^{\sigma} - y^{\sigma}$

degree = σ

Case (ii) : $p < 0$ and $a \equiv 0\ [p]$

$$\begin{pmatrix} (k+1)r & b \\ kr & b+r \end{pmatrix}, \text{ with } r, k, b > 0 \ .$$

$$[z - K(x,y)]\,Y^{r(k+1)+b} + \sum_{i=0}^{k} \binom{k}{i} \frac{(x^{-r} - y^{-r})^i}{r(k+1-i)+b}\, Y^{ir} = 0$$

$$\text{degree} = \sigma$$

Remark. For $k = 1$, $r = a$ we retrieve [M. 2012, *LATIN*]

$$\begin{pmatrix} 2a & b \\ a & a+b \end{pmatrix}$$

Case (iii) : $p \geq 2$, $a \equiv 1$ $[p]$ and $b \equiv -1$ $[p]$

$$\begin{pmatrix} kr+1-r & r\ell-1+r \\ kr+1 & r\ell-1 \end{pmatrix}, \text{ with } k,\ell > 0 \text{ and } r > 1 \ .$$

$$\frac{P(Y^r)}{(Y^r + C)^{k+1/r-1}\, Y^{r\ell-1}\, C^{\ell+k}} = z - K(x,y)$$

where P is a polynomial of degree $k+\ell-1$.

and $C = x^r - y^r$.

Let $Q(Y) := P(Y^r)$. Then,

$$Y(Y^r + C)Q'(Y) + [(r-1-kr)Y^r + (1-r\ell)(Y^r + C)]\, Q(Y) = C^{k+\ell}$$

$$\text{degree} = r(rk+1-r) + r(r\ell-1) = r^2(k+\ell-1)$$

Conclusion

- Conjecture: There is no other algebraic urn.
- Asymptotic properties

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Bon ALEA à tous !